Classical solution of the wave equation

M. N. Sergeenko

The National Academy of Sciences of Belarus, Institute of Physics
Minsk 220072, Belarus,
Homel State University, Homel 246699, Belarus and
Department of Physics, University of Illinois at Chicago, Illinois 60607, USA

Abstract

The classical limit of wave quantum mechanics is analyzed. It is shown that the general requirements of continuity and finiteness to the solution $\psi(x) = Ae^{i\phi(x)} + Be^{-i\phi(x)}$, where $\phi(x) = \frac{1}{\hbar}W(x)$ and $W(x)$ is the reduced classical action of the physical system, result in the asymptote of the exact solution and general quantization condition for $W(x)$, which yields the exact eigenvalues of the system.

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1. Introduction. One of the fundamental principles of quantum mechanics is the Bohr's correspondence principle. This principle has been used at the stage of creation of quantum theory and to derive the wave equation. Besides, this same principle results in the simplest solution of the Schrödinger equation.

There is a mathematical realization of the correspondence principle known as the quasiclassical approximation [1] applicable in the case when the de Broglie wavelength, $\lambda = h/p$ ($h = 2\pi\hbar$), is a slowly changing function of position. One of advantages of the quasiclassical approach is that we can calculate a tunneling effect that is beyond the usual perturbative methods. To any finite order in perturbative theory, we will never see any of these nontrivial nonperturbative effects. The quasiclassical approach also naturally leads to the conception of instantons [2, 3] which have proved to be a powerful tool to probe the nonperturbative regime of gauge theory. QCD instantons force us to re-examine the whole question of CP violation [4].

As well known the exact eigenvalues can be defined with the help of the asymptotic solution of the wave equation. In Refs. [5, 6] the correspondence principle has been used to derive the semiclassical wave equation appropriate in the quasiclassical region. It was shown that solution of this equation by the standard WKB method yields the exact eigenvalues for *all* known solvable problems in quantum mechanics.

In this letter, we scrutinize the classical limit of the Schrödinger wave mechanics for conservative physical problems. The general solution $\psi_0(x)$ of the wave equation and the quantization condition is written in terms of the classical action. The wave function (*wf*) in the whole region is built with the use of requirements of continuity and finiteness for $\psi_0(x)$ in the whole region.

2. Connection formulas. Consider the Schrödinger equation for the arbitrary potential $V(x)$ in one dimension[1],

[1] Multidimensional separable problems can be treated analogously. In this case, separation should be performed with the help of the correspondence principle between classical and quantum-mechanical quantities [7].

$$\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)=\left[P^2-U(x)\right]\psi(x)\ ,\tag{1}$$

where $P^2=2mE$ and $U(x)=2mV(x)$. As well known, in the classical limit the general solution of Eq. (1) can be written in the form,

$$\psi_0(x)=Ae^{i\phi(x)}+Be^{-i\phi(x)},\tag{2}$$

where A and B are the arbitrary constants and

$$\phi(x)=\frac{1}{\hbar}W(x)\equiv\frac{1}{\hbar}\int^x\sqrt{P^2-U(x)}dx\tag{3}$$

is the dimensionless phase variable. The function $W(x)$ (the reduced classical action of the system[2]) satisfies the classical Hamilton-Jacobi equation, $(\frac{dW}{dx})^2=P^2-U(x)$, where $\frac{dW}{dx}\equiv p(x)$ is the generalized momentum.

Function (2) can be treated as the "classical" solution of the Schrödinger equation (1). This solution has several remarkable features. The most important is that the solution (2) has no divergence at the classical turning points (TP) given by $E-V=0$. Another interesting feature is that, for discrete spectrum, the requirements of continuity and finiteness for $\psi_0(x)$ in the whole region result in the asymptotic solution and general quantization condition for the classical action of the system.

Solution in quantum mechanics must be continuous finite function in the whole region. To build the *wf* in the whole region we need to connect (in turning points) solution in the classically allowed region with solution in the classically inaccessible region. Consider the general solution (2) in the region of the classical TP x_k. In the classically allowed region [given by $E\geq V$] solution (2) can be written as

$$\psi_0^I(\phi)=A_ke^{i(\phi-\phi_k)}+B_ke^{-i(\phi-\phi_k)}.\tag{4}$$

In the classically inaccessible region [where $E<V$] the general solution (2) is

$$\psi_0^{II}(\phi)=C_ke^{-\phi+\phi_k}+D_ke^{\phi-\phi_k}.\tag{5}$$

The functions (4) and (5) must satisfy the continuity conditions, $\psi_0^I(\phi_k)=\psi_0^{II}(\phi_k)$ and $d[\psi_0^I(\phi_k)]/dx=d[\psi_0^{II}(\phi_k)]/dx$, at $\phi=\phi_k$ [3].

Matching the functions (4) and (5) and their first derivatives at the TP x_k gives

$$\left\{\begin{array}{l}A_k+B_k=C_k+D_k,\\ iA_k-iB_k=-C_k+D_k.\end{array}\right.\tag{6}$$

This yields

$$\left\{\begin{array}{l}A_k=\frac{1}{\sqrt{2}}\left(C_ke^{i\pi/4}+D_ke^{-i\pi/4}\right),\\ B_k=\frac{1}{\sqrt{2}}\left(C_ke^{-i\pi/4}+D_ke^{i\pi/4}\right).\end{array}\right.\tag{7}$$

[2]For conservative classical system, the total action is $S_0(t,x)=-Et+W(x)$.

[3] Note, the well-known WKB approximation cannot be used in the region near the TP, because when $E=V$, the conditions for its applicability break down.

The connection formulas (7) supply the continuous transition of the general solution (4) into solution (5) at the TP x_k.

3. The "classical" *wf*. Quantization. Now we can build the "classical" *wf* for a given physical problem. To build the *wf*, we need to choose the boundary conditions for the problem. Consider first the two-turning-point (2TP) problem. For the 2TP problem, the whole interval $(-\infty, \infty)$ is divided by the TP x_1 and x_2 into three regions, $-\infty < x < x_1$ (I), $x_1 \le x \le x_2$ (II), and $x_2 < x < \infty$ (III). The classically allowed region is given by the interval II.

In the classically inaccessible regions I and III we choose the exponentially decaying solutions, i.e., $\psi_0^I(\phi) = D_1 e^{\phi - \phi_1}$ left from the TP x_1 (we put $C_1 = 0$), and $\psi_0^{III}(\phi) = C_2 e^{-\phi + \phi_2}$ right from the TP x_2 (here we put $D_2 = 0$). Then, in the classically allowed region II, right from the TP x_1 we have, from (7), $A_1 = \frac{D_1}{\sqrt{2}} e^{-i\pi/4}$ and $B_1 = \frac{D_1}{\sqrt{2}} e^{i\pi/4}$, and solution (4) takes the form,

$$\psi_0^{II}(\phi) = \sqrt{2} D_1 \cos\left(\phi - \phi_1 - \frac{\pi}{4}\right), \tag{8}$$

and left from the TP x_2 [here $A_2 = \frac{C_2}{\sqrt{2}} e^{i\pi/4}$ and $B_2 = \frac{C_2}{\sqrt{2}} e^{-i\pi/4}$] solution (4) is

$$\psi_0^{II}(\phi) = \sqrt{2} C_2 \cos\left(\phi - \phi_2 + \frac{\pi}{4}\right), \tag{9}$$

where $\phi_1 = \phi(x_1)$ and $\phi_2 = \phi(x_2)$. We see that the superposition of two plane waves (2) in the phase space results in the standing wave given by Eqs. (8) and (9).

Functions (8) and (9) should coincide in each point of the interval $[x_1, x_2]$. Putting $\phi = \phi_2$ we have, from Eqs. (8) and (9),

$$D_1 \cos\left(\phi_2 - \phi_1 - \frac{\pi}{4}\right) = C_2 \cos\frac{\pi}{4}\ . \tag{10}$$

This equation is valid if

$$\phi_2 - \phi_1 - \frac{\pi}{4} = \frac{\pi}{4} + \pi n, \quad n = 0, 1, 2, \dots \tag{11}$$

and $D_1 = (-1)^n C_2$. Equation (11) is the condition of the existence of continuous finite solution in the whole region. This condition being, at the same time, quantization condition. Taking into account the notation (3), we have, from Eq. (11),

$$\int_{x_1}^{x_2} \sqrt{P^2 - U(x)}dx = \pi\hbar\left(n + \frac{1}{2}\right). \tag{12}$$

Condition (12) solves the 2TP eigenvalue problem given by Eq. (1). Note, we have obtained Eq. (12) by product from requirements of a smooth transition from oscillating solution (4) to the exponentially decaying solutions in the classically inaccessible regions. The quantization condition (12) reproduces the exact eigenvalues for *all* known 2TP problems in quantum mechanics (see Refs. [5, 6]).

Combining the above results, we can write the finite continuous solution in the whole region (the "classical" wave function),

$$\psi_0[\phi(x)] = C \begin{cases} \frac{1}{\sqrt{2}} e^{\phi(x)-\phi_1}, & x < x_1, \\ \cos[\phi(x) - \phi_1 - \frac{\pi}{4}], & x_1 \le x \le x_2, \\ \frac{(-1)^n}{\sqrt{2}} e^{-\phi(x)+\phi_2}, & x > x_2. \end{cases} \tag{13}$$

Oscillating part of solution (13) corresponds to the main term of the asymptotic series in theory of the second-order differential equations. In quantum mechanics, the oscillating part of Eq. (13) gives the asymptote of the exact solution of the Schrödinger equation.

An important consequence of the "classical" solution (2) is conservation not only energy E, but also momentum. In Eq. (1), $P^2 = 2mE$ is a constant and, as E, it can take only discrete values. Substituting (2) into Eq. (1), we have

$$\left(\frac{dW}{dx}\right)^2 - i\hbar \frac{d}{dx}\left(\frac{dW}{dx}\right) = P^2 - U(x). \tag{14}$$

So far, as $(\frac{dW}{dx})^2 = P^2 - U(x)$, we obtain the constraint,

$$\hbar \frac{d}{dx}\left(\frac{dW}{dx}\right) = 0. \tag{15}$$

The constant $\hbar$ is a small value, which is used as the expansion parameter in the quasiclassical approximation. The constraint (15) can be achieved not only by taking the limit $\hbar \to 0$ (that leads to the classical mechanics), but also by assuming that[4]

$$\frac{dW}{dx} \simeq \text{const}. \tag{16}$$

This implies the adiabatically slow alteration of the momentum $p(x) \equiv W'(x)$. In this case, $\hbar W''(x)$ is a small value of the second order and can be neglected. Note, the constraint (16) supplies the Hermiticity of the operator $\hat{p}^2 = (-i\hbar\frac{d}{dx})^2$ in Eq. (1) [5]; in order for the operator $\hat{p}^2$ to be Hermitian the expression $W'^2 - i\hbar W''$ should be real.

Underline, the potential $V(x)$ in Eq. (1) is not a constant. The constraint (16) is the requirement to the *final* solution we build from the general expression (2); it is valid only for certain values of $P_n^2 = \langle p^2(x)\rangle$, which correspond to the discrete values of the action $W(x)$ given by Eq. (12). This is a specific requirement for the allowed motions in quantum mechanics: for the conservative quantum-mechanical systems, the particle momentum in stationary states is the *integral of motion*.

According to Bohr's postulates, 1) electrons in atomic orbits are in stationary states, i.e. do not radiate, despite their acceleration, and 2) electrons can take discontinuous transitions from one allowed orbit to another. However, the postulate is not an answer to the question: why the accelerated electrons do not radiate?

Acceleration means change in momentum. As we have shown above, particles in the stationary states are *not* accelerated, because the momentum is the integral of motion. The fact the momentum is a constant value means the electrons in stationary states move like free particles. In these states, the momentum eigenvalues and the energy eigenvalues

[4] The well known WKB approximation is based on the condition $W'^2 \gg |\hbar W''|$ that implies that the momentum $p(x) = W'(x)$ is large enough.

are connected with the help of the equality $P_n^2 = 2mE_n$ for free particles. Such kind of wave motion is described by the standing waves.

The oscillating part of the *wf* (13) is in agreement with the asymptote of the corresponding exact solution of Eq. (1) and can be written in the form of a standing wave. Integrating (16), we obtain $W(x,n) = P_n x + \mathrm{const}$ that gives, for the function $\psi_0^{II}(x)$,

$$\psi_0^{II}(x) = C_n \cos\left(k_n x + \frac{\pi}{2}n\right), \tag{17}$$

where $k_n = P_n/\hbar$. The normalization coefficient, $C_n = \sqrt{2k_n/[\pi(n+\frac{1}{2})+1]}$, is calculated from the normalization condition $\int_{-\infty}^{\infty} |\psi_0(x)|^2 = 1$.

In Eq. (17), we have took into account the fact that, in the stationary states, the phase-space integral (3) at the TP x_1 and x_2 is $\phi_1 = -\frac{\pi}{2}(n+\frac{1}{2})$ and $\phi_2 = \frac{\pi}{2}(n+\frac{1}{2})$, respectively, [so that $\phi_2 - \phi_1 = \pi(n+\frac{1}{2})$] [5, 7], i.e. it depends on quantum number and does not depend on the form of the potential. This form of ϕ_1 and ϕ_2 guaranties that the eigenfunctions are necessarily either symmetrical ($n = 0, 2, 4, \ldots$) or antisymmetrical ($n = 1, 3, 5, \ldots$). Solution (17) describes free motion of a particle-wave in the enclosure (the enclosure being the interaction potential). Therefore, in bound state region, the interaction of a particle-wave with the potential reduces to reflection of the wave by the walls of the potential.

The "classical" solution (2) is general for all types of problems and allows to solve multi-turning-point problems (MTP, $M > 2$), i.e. a class of the "insoluble" problems, which cannot be solved by standard methods. In the complex plane, the 2TP problem has one cut between turning points x_1 and x_2, and the phase-space integral (12) can be written as the contour integral about the cut. The MTP problems contain (in general case) bound state regions and the potential barriers, i.e. several cuts. The corresponding contour should enclose all cuts. The "classical" *wf* in the whole region can be built similar to the 2TP problem with the help of the same connection formulas (7).

Let the problem has ν cuts. Then the integral about the contour C can be written as a sum of contour integrals about each of the cut, where each term of the sum is the 2TP problem. Then, the 2ν TP quantization condition can be written as [6, 7]

$$\oint_C \sqrt{P^2 - U(z)}dz = 2\pi\hbar\left(N + \frac{\mu}{4}\right), \tag{18}$$

where $N = \sum_{k=1}^{\nu} n_k$ is the total number of zeroes of the *wf* on the ν cuts and $\mu = 2\nu$ is the number of turning points, i.e. number of reflections of the *wf* by the walls of the potential (Maslov's index [8]).

4. Relativistic Cornell problem, $V(r) = -\frac{\tilde{\alpha}}{r} + \kappa r$**.** To demonstrate efficiency of the "classical" solution, let us obtain the exact energy eigenvalues for the famous funnel type potential (Cornell potential) [9]. This potential is one of a special interest in high-energy hadron physics, quarkonium physics, and quark potential models. Its parameters are directly related to basic physical quantities of hadrons: the universal Regge slope $\alpha' \simeq 0.9\,(\mathrm{GeV}/c)^{-2}$ of light mesons and one-gluon-exchange coupling strength, α_s, of heavy quarkonia.

In relativistic theory, the Cornell potential represents the so-called "insoluble" 4TP problem. The "classical" solution and quantization condition derived above give the

asymptote of the exact solution and yield the exact energy spectrum for the potential. A series of examples has been considered elsewhere [5, 6, 7], where the exact energy spectrum of quantum systems was reproduced from quasiclassical solution of the semiclassical wave equation [5, 6].

For the system of two particles of equal masses, the relativistic radial semiclassical wave equation for the Cornell potential is ($\hbar = c = 1$) [6]

$$\left(-i\frac{d}{dr}\right)^2 \tilde{R}(r) = \left[\frac{E^2}{4} - \left(m - \frac{\tilde{\alpha}}{r} + \kappa r\right)^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right] \tilde{R}(r), \tag{19}$$

where $\tilde{\alpha} = \frac{4}{3}\alpha_s$. The quantization condition (18) appropriate to Eq. (19) is $I = \oint_C [\frac{1}{4}E^2 - (m - \tilde{\alpha}/r + \kappa r)^2 - (l+\frac{1}{2})^2/r^2]^{\frac{1}{2}} dr = 4\pi\left(n_r + \frac{1}{2}\right)$. To calculate this integral, we use the method of stereographical projection that gives $I = I_0 + I_\infty$, where $I_0 = -2\pi\Lambda$ and $\Lambda = [(l+\frac{1}{2})^2 + \tilde{\alpha}^2]^{\frac{1}{2}}$. The integral I_∞ is calculated with the help of the replacement of variable, $z = \frac{1}{r}$, that gives $I_\infty = 2\pi[E^2/8\kappa + \tilde{\alpha}]$. Therefore, for E_n^2, we obtain

$$E_n^2 = 8\kappa\left[2\left(n_r + \frac{1}{2}\right) + \Lambda - \tilde{\alpha}\right]. \tag{20}$$

This is the exact result for the Cornell potential.

It is an experimental fact that the dependence $E_n^2(l)$ is linear for light mesons. At present, the best way to reproduce the experimental masses of particles is to rescale the entire spectrum for the linear potential, $E_n^2 = 8\kappa(2n_r + l + \frac{3}{2})$, assuming that the masses of the mesons are expressed by the relation $M_n^2 = E_n^2 - C^2$, where C is an additional free parameter. At the same time, Eq. (20) does not require any additional free parameter. We obtain the necessary shift with the help of the term $-8\kappa\tilde{\alpha}$ which is the result of interference of the Coulomb and linear terms of the interquark potential. The formula describes the light meson trajectories with the accuracy $\simeq 1\%$.

5. Discussion and Conclusion. In the classical limit $\hbar \to 0$, the quantum-mechanical action $S(t,x)$ reduces to the classical action $S_0(t,x)$, i.e. $S(t,x) = S_0(t,x)$. Then, for conservative systems, we can write $S(t,x) = -Et + W(x)$, where $W(x)$ is the reduced classical action, and the general solution $\psi_0(x)$ of the wave equation can be written in the form of superposition of two waves, $\exp[\pm iW(x)/\hbar]$.

Using the general requirements of continuity and finiteness for $\psi_0(x)$ and $\psi_0'(x)$, we have derived simple connection formulas that allowed us to build the "classical" *wf* in the whole region and the corresponding quantization condition for the classical action. If well-known quasiclassical approximation is applicable for a distance from the turning point satisfying the condition $|x - x_0| \gg \lambda/4\pi$ and potentials satisfying the semiclassical condition, $m\hbar V' \ll 2m[E - V]^{3/2}$, the "classical" solution considered in this Letter is applicable in the whole region and for any separable potentials.

The final solution has written in terms of elementary functions and corresponds to the main term of the asymptotic series in the theory of the second-order differential equations. We have observed that, for the conservative systems, not only energy, but also momentum is the integral of motion. This means that particles in stationary states move like free particles-waves in enclosures. This has allowed us to write the "classical" solution in the form of the standing wave, which describes free finite motion of particles-waves in enclosures.

There is a simple connection of the "classical" solution considered here with the Feynman path integrals: this solution corresponds to the path of "minimum" action. The "classical" solution obtained corresponds to the classical path.

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